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ON THE ASYMPTOTIC PROPERTIES OF A PRIORI MINIMAX ESTIMATES*

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Properties are studied of the a priori minimax estimates /1,2/ of an unknown initial state of a linear dynamic object under the condition that the measurements are taken at discrete instants, while noise exists only in the measurement channel and is modelled as an independent random variable. The paper continues the investigations begun in /3/.

1. Preliminary remarks. Assume that the process

$$y_k = Bx(t_k) + w_k \tag{1.1}$$

where $x(t)$ is a linear system's phase vector, is observed at discrete instants of time $t_k = k\tau,\ k = 0, 1, 2, \ldots\ (\tau > 0)$

$$x^{\cdot} = Ax,\ t \geqslant 0,\ x(0) = z \in R^n \tag{1.2}$$

Here A and B are constant $(n \times n)$- and $(m \times n)$-matrices, $w_k,\ k = 0, 1, \ldots,$ are independent random variables prescribed on the space of elementary events $\{\Omega, \Sigma, P\}$ in $\{R^m, \Delta\}$, Δ is a Borel σ-algebra of sets in R^m. The initial state z is unknown. Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of matrix A of multiplicities $k_1, \ldots, k_r$, respectively, and $h_{ij},\ j = 1, 2, \ldots, k_i$, be vectors of a series relative to the matrix A with eigenvalues λ_i, so that /4/

$$Ah_{i1} = \lambda_i h_{i1}, \quad Ah_{i2} = \lambda_i h_{i2} + h_{i1}, \ldots,$$
$$Ah_{ik_i} = \lambda_i h_{ik_i} + h_{ik_i-1}, \quad i = 1, 2, \ldots, r$$

We accept that

$$\operatorname{Re}\lambda_1 \leqslant \operatorname{Re}\lambda_2 \leqslant \ldots \leqslant \operatorname{Re}\lambda_u < 0 = \operatorname{Re}\lambda_{u+1} = \ldots = \operatorname{Re}\lambda_{u+p} <$$
$$\operatorname{Re}\lambda_{u+p+1} \leqslant \ldots \leqslant \operatorname{Re}\lambda_r$$

and we denote the sets

$$\Psi = \{\psi \in R^n : \|\psi\| = 1,\ \psi h_{ij} = 0,\ j = 1, 2, \ldots, k_i,$$
$$i = 1, 2, \ldots, u\},\ \Psi' = \{\psi \in \Psi : \psi = \alpha B + a\psi',$$
$$\alpha \in R^m,\ a \in R',\ \psi' h_{i1} = 0,\ i = u+1, u+2, \ldots, u+p\} \tag{1.3}$$

by Ψ, Ψ'. By lower-case Greek letters we denote row-vectors of appropriate dimensions, in contrast to column-vectors which we denote by lower-case Latin letters.

An estimate $z(\psi, N)$ of the scalar quantity $\psi z,\ \psi \in R^n,\ \|\psi\| = 1$, from N observations of form (1.1) can be realized as a linear functional

$$z(\psi, N) = \varphi[y^N] = \sum_{i=0}^{N} \varphi_i y_i, \quad y^N = (y_0, \ldots, y_N) \tag{1.4}$$

As was shown in /1/, the operation $\varphi[\cdot]$ is best in the minimax sense under the assumption that the possible domain of noise measurements is estimated by the inequality $\rho[w^N] \leqslant 1$, where $w^N = (w_0, \ldots w_N)$ and $\rho[\cdot]$ is the norm in the (mN)-dimensional space $\{w^N\}$, and is determined as the solution of the following problem (the moment problem):

$$\rho^*[\varphi] \to \inf; \quad \sum_{i=0}^{N} \varphi_i B\Phi(i\tau) = \psi \tag{1.5}$$

Here $\Phi(t)$ is the fundamental matrix of (1.2) and $\rho^*[\cdot]$ is the adjoint norm, i.e., the norm introduced in the usual manner in the space of linear functionals over $\{w^N, \rho[\cdot]\}$. The following norms ($\|\cdot\|$ is the Euclidean norm in R^m):

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$$\rho_q[w^N]=\Big(\sum_{k=0}^{N}\|w_k\|^q\Big)^{1/q},\qquad q=1,2$$

$$\rho_\infty[w^N]=\max\{\|w_k\|:0\leqslant k\leqslant N\}$$

are often used as $\rho[\cdot]$. By $z_q(\psi, N)$ we denote the minimax estimate corresponding to the following norm $\rho_q[\cdot]$, $q=1, 2, \infty$. Further on we study the asymptotic properties of $z_q(\psi, N)$ as $N\to\infty$.

2. Fundamental statements. Theorem 1. Let the pair (A, B) be observable, $\mathbf{M}[w_k]\equiv 0$, $\mathbf{M}[w_k w_k^*]\equiv d^2 I$ (I is the unit matrix, the asterisk denote transposition). Then a $\tau>0$ exists such that for all $\psi\in\Psi$, $z_q(\psi, N)$, $q=1, 2$, is an unbiased and consistent estimate, i.e., $\mathbf{M}[z_q(\psi, N)]=\psi z$, and for any $\varepsilon>0$ we have $P\{|z_q(\psi, N)-\psi z|\geqslant\varepsilon\}\to 0$ as $N\to\infty$.

Generally speaking, the analogous statement for $q=\infty$ is invalid. However, if δ_* is the value of the lower bound in (1.5), then the next theorem holds when $\rho[\cdot]=\rho_\infty[\cdot]$.

Theorem 2. Let system (A, B) be observable, $\mathbf{M}[w_k]\equiv 0$ and $\mathbf{M}[w_k w_k^*]$ be bounded for $k=0, 1, \ldots$. Then a $\tau>0$ exists and for all $\psi\in\Psi'$, $\varepsilon>0$, we can find a functional φ_ε such that $\rho_\infty^*[\varphi_\varepsilon]<\delta_*+\varepsilon$ and estimate (1.4) is unbiased and strongly consistent, i.e., with probability one

$$z_\varepsilon(\psi, N)=\varphi_\varepsilon[y^N]\to\psi z\quad\text{as}\quad N\to\infty$$

The minimax estimates are guaranteed and are constructed during calculations on the worst realizations of the noise, not using their statistical characteristics. From Theorems 1 and 2 it follow that if the noise is random, they nevertheless possess satisfactory properties (unbiasedness, consistency). We note that the set Ψ covers the most interesting directions, since if $z=c_1\psi_1^*+c_2\psi_2^*$, $\psi_1\in\Psi$ and $\psi_2\psi^*=0$ for $\psi\in\Psi$, then $\Phi(t)z=c_1\Phi(t)\psi_1^*+\delta(t)$, $\|\delta(t)\|\leqslant C\exp(\operatorname{Re}\lambda_u t)\to 0$ as $t\to\infty$, $C=\text{const}$. From the observability condition it follows that for all $i=1, 2, \ldots, r$ there exists a row β of matrix B such that $\beta h_{i1}\neq 0$.

3. Proofs of the fundamental statements. Here, for brevity, Theorems 1 and 2 are not proved in detial. Below, only a proof plan is presented in the form of a sequence of statements. For simplicity we assume that $m=1$, i.e., matrix B consists of one row β. The passage to the general case can be effected by a decomposition of the space into a direct sum of observability subspaces connected with the rows of matrix B.

1°. Let the pair (A, β) be observable. Consider the system of equations

$$|\,\beta\Phi((j-k)\tau)g_k=\delta_{jk};\quad j=1, 2, \ldots, n,\ g_k\in R^n \tag{3.1}$$

where δ_{jk} is the Kronecker symbol, k is fixed. There exists $\tau_*>0$ such that system (3.1) is solvable when $\tau=l_s\tau_*$, $s=1, 2, \ldots$, and its solution admits of the representation

$$g_k=\sum_{i=1}^{r}\sum_{j=1}^{k_i}c_{ij}^k(\tau)h_{ij};\ |c_{ij}^k(l_s\tau_*)|\leqslant (l_s\tau_*)^{-(j-1)}G,\quad G=\text{const},\quad l_s/s\leqslant L<\infty,\quad s=1, 2, \ldots \tag{3.2}$$

Representation (3.2) can be obtained directly by inversion of the matrix of system (3.1), reduced to the new variables

$$c'_{ij}=\sum_{k=j}^{k_i}\frac{c_{ik}(\beta h_{ik-j+1})}{(j-1)!},\quad j=1, 2, \ldots, k_i,\quad i=1, 2, \ldots, r$$

We make use of the indicated values of τ_* and l_s, $s=1, 2, \ldots$.

2°. There exist $M>0$ and a partitioning $\{J_k\}$ of set $\{0, 1, \ldots, N\}$ into collections J_k, $k=1, 2, \ldots, K(N)$, of n indices each, such that $J_k\cap J_i=\varphi$, $k\neq i$, and the system of equations

$$\sum_{i\in J_k}\varphi_i\beta\Phi(i\tau_*)=\psi,\ \psi\in\Psi \tag{3.3}$$

has the solution

$$\varphi_i\colon |\varphi_i|\leqslant M,\quad i\in J_k,\quad k=1, 2, \ldots, K(N),\ N=1, 2, \ldots$$

Here the partitioning $\{J_k\}$ can be chosen such that $N/K(N)<\infty$ as $N\to\infty$. To prove this assertion we consider the collections

$$J_k=\{i=k+(j-1)l_K,\ j=1, 2, \ldots, n\},\quad k=0, 1, \ldots, K$$

and seek the solution of (3.3) as

$$\varphi_i = \psi\Phi(-i\tau_*)\, g_i \tag{3.4}$$

This leads to the equation system (3.1) in g_i with $\tau = l_K\tau_*$. An upper bound for $|\varphi_i|$ thus follows from relations (3.2), and to complete the proof it remains to note that $K(N) \geqslant \left[\frac{N}{1+nL}\right]$ for the collections J_k of the form being examined.

3°. Proof of Theorem 1. Let

$$z_1(\psi, N) = \sum_{i=1}^{N} \varphi_i y_i = \varphi[y^N]$$

Since $\rho_1^*[\cdot] = \rho_\infty[\cdot]$, the inequality

$$|\varphi_i| \leqslant M(K(N))^{-1}, \quad i = 0, 1, \ldots, N$$
$$\mathbf{M}[|z_1(\psi, N) - \psi z|^2] \leqslant M^2 d^2 N (K(N))^{-2}$$

follows from 2°. The estimate $z_2(\psi, N)$ can be written out in explicit form /1/. Under the theorem's hypotheses it coincides with the estimate from the least squares method in the general nondegenerate linear Gauss—Markov model /5-7/

$$\mathbf{M}[|z_2(\psi, N) - \psi z|^2] = d^2\psi\Big(\sum_{i=0}^{N} \Phi^*(i\tau_*)\beta^*\beta\Phi(i\tau_*)\Big)^{-1}\psi^* \leqslant M^2 n (K(N))^{-1}$$

The last inequality follows from statement 2°, since the relations

$$M^2 n \geqslant \sum_{i\in J_k} |\varphi_i|^2 = \psi\Big(\sum_{i\in J_k} \Phi^*(i\tau_*)\beta^*\beta\Phi(i\tau_*)\Big)^{-1}\psi^*$$

are valid for the solutions of (3.3). Thus, the assertion of Theorem 1 follows from the Chebyshev inequality. The unbiasedness of the estimates is a simple corollary of the unbiasedness of the minimax estimates /1/, expressed by the second of relations (1.5).

4°. Let

$$u + p \geqslant 1, \quad \psi = \beta(\|\beta\|)^{-1}, \quad \rho[\cdot] = \rho_\infty[\cdot]$$

Then the relations

$$\varphi_0^\circ = (\|\beta\|)^{-1}, \quad \varphi_i^\circ = 0, \; i = 1, 2, \ldots, N \tag{3.5}$$

determine the solution of (1.5). Indeed

$$\sum_{i=0}^{N} \varphi_i^\circ \beta\Phi(i\tau_*) = \psi$$

Assume that φ_i, $i = 0, 1, \ldots, N$, is a solution of (1.5). Then (see /1/)

$$\rho_\infty^*[\varphi] = \sum_{i=0}^{n} |\varphi_i| \geqslant \mu^{-1}, \qquad \mu = \inf_z \{\rho_\infty[(y^\circ)^N] : \psi z = 1\}$$

where y_i°, $i = 0, 1, \ldots, N$ is an ideal noisefree signal. Allowing for (1.1) and (1.2), we obtain

$$\mu = \inf_z \{\max_{0\leqslant i\leqslant N} |\beta\Phi(i\tau)z| : \psi z = 1\} =$$
$$\|\beta\| \inf_{c_{ij}} \Big\{\max_{0\leqslant i\leqslant N} \Big|\sum_{k=1}^{r} \exp(\lambda_k i\tau_*) \sum_{j=1}^{k_i} c_{kj} \times$$
$$\sum_{l=1}^{k_j} \frac{(i\tau_*)^{j-l}}{(j-l)!}\psi h_{kl}\Big| : \sum_{k=1}^{r}\sum_{j=1}^{k_i} c_{kj}\psi h_{kj} = 1\Big\} \leqslant \|\beta\|$$

5°. Let

$$\psi = a_1(\|\beta\|)^{-1}\beta + a_2\psi' \in \Psi', \; \rho[\cdot] = \rho_\infty[\cdot]$$

and $\varphi_i(T, N)$, $i = 0, 1, \ldots, N$ be the solution of (1.5) under the additional assumption $\varphi_0 = \varphi_1 = \ldots = \varphi_T = 0$. Then

$$\sum_{i=0}^{N} |\varphi_i(T, N)| = \sum_{i=T+1}^{N} |\varphi_i(T, N)| \to |a_1|(\|\beta\|)^{-1}, \quad N \to \infty$$

Indeed, from the proposition in 2° it follows that system (3.3) is solvable for each of the collections

$$J_{ks} = \{t = k + (j-1)\, l_s,\; j = 1, 2, \ldots, n\},\; k, s = 1, 2, \ldots$$

Choosing $k \geqslant T$ and allowing for (3.2) and (3.4), for $N \geqslant k + nl_s$ we obtain

$$|a_1|(\|\beta\|)^{-1} \leqslant \sum_{i=0}^{N} |\varphi_i(T, N)| \leqslant |a_1| \sum_{t \in J_{ks}} \left| \sum_{i=1}^{r} \exp(-\lambda_i t\tau_*) \sum_{j=1}^{k_i} c_{ij}(l_s\tau_*) \times \right.$$
$$\left. \sum_{m=1}^{j} \frac{(t\tau_*)^{j-m}}{(j-m)!} (\|\beta\|)^{-1}\beta h_{im} \right| + |a_2| \sum_{t \in J_{ks}} \left| \sum_{i=1}^{r} \exp(-\lambda_i t\tau_*) \times \right.$$
$$\left. \sum_{j=1}^{k_i} c_{ij}(l_s\tau_*) \sum_{m=1}^{j} \frac{(t\tau_*)^{j-m}}{(j-m)!} \psi' h_{im} \right|$$

Using (1.3) and (3.2), we can prove that as $s \to \infty$ the first of the sums in the last expression tends to the quantity $(\|\beta\|)^{-1}$, while the second tends to zero. Consequently, statement 5° is valid.

6°. Proof of Theorem 2. Relying on the preceding statement, we choose the sequence N_k, $k = 1, 2, \ldots,$ from the condition

$$\sum_{i=N_k+1}^{N_{k+1}} |\varphi_i(N_k, N_{k+1})| < |a_1|(\|\beta\|)^{-1} + \varepsilon$$

It is well known that when $\rho[\cdot] = \rho_\infty[\cdot]$ the solution of (1.5) is reached on a finite collection of indices. In the case at hand we can take no more than n indices /1,2/. By φ_{ki}, $i = 1, 2, \ldots, n$, we denote nonzero $\varphi_i(N_k, N_{k+1})$ and we define

$$\varphi_\varepsilon[y^N] = \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{n} \varphi_{ki} y_{ki}, \quad N_{K+1} \leqslant N \leqslant N_{K+2}$$

By virtue of statements 4° and 5°

$$\rho_\infty^*[\varphi_\varepsilon] = \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{n} |\varphi_{ki}| < \delta_* + \varepsilon$$

Theorem 2 follows from the unbiasedness of the minimax estimate and the strong law of large numbers /8/.

4. Examples. We assume that w_k, $k = 0, 1, \ldots,$ are independent like-distributed random variables with zero mean and bounded variance.

1°. Let $n = 1, A = 0, B = 1$. The estimates considered in the paper are determined by the equality

$$z_q(\psi, N) = \frac{1}{(N+1)} \sum_{k=0}^{N} y_k = z + \frac{1}{N+1} \sum_{k=0}^{N} w_k$$

(the solution is not unique when $q = \infty$).

2°. We consider the small oscillations of a mathematical pendulum. In this case

$$A = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|, \quad \beta = \|1, 0\|, \quad \beta\Phi(k\tau) = \|\cos k\tau, \sin k\tau\|$$

If $\psi = \|\psi^1, \psi^2\|$ and $\tau = \tau_* = {}^1/_2\pi$, then

$$z_1(\psi, 2K-1) = \frac{1}{K} \sum_{i=0}^{K-1} ((-1)^i \psi^2 y_{2i+1} + (-1)^{i+1} \psi^1 w_{2i}) =$$
$$\psi z + \frac{1}{K} \sum_{i=0}^{K-1} ((-1)^i \psi^2 w_{2i+1} + (-1)^{i+1} \psi^1 y_{2i})$$

3°. For a uniform motion along a straight line

$$A = \left\| \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \right\|, \quad \beta = \|1, 0\|, \quad \beta\Phi(t) = \|1, t\|, \quad \tau = 1$$

we obtain the following expression for the least squares estimate:

$$z_2(\psi, N) = \sum_{t=0}^{N} \varphi_t y_t = \sum_{t=0}^{N} \psi R(N) \Phi^*(t) \beta^* y_t$$

$$R(N) = \left(\sum_{t=0}^{N} \Phi^*(t) \beta^* \beta \Phi(t)\right)^{-1} = \begin{Vmatrix} \frac{2(2N+1)}{(N+1)(N+2)} & -\frac{6}{(N+1)(N+2)} \\ -\frac{6}{(N+1)(N+2)} & \frac{12}{N(N+1)(N+2)} \end{Vmatrix}$$

The minimax estimates of the initial position $(\psi = \psi_1 = \| 1, 0 \|)$ and velocity $(\psi = \psi_2 = \| 0, 1 \|)$ when $q = \infty$ are determined by the relations

$$z_\infty(\psi_1, N) = \varphi^1[y^N] = y_0, \quad z_\infty(\psi_2, N) = \varphi^2[y^N] = \frac{y_N - y_0}{N}$$

and are not consistent. Let $\varepsilon > 0$. We define sequences of positive integers m_i and k_i, $i = 1, 2$, $m, k = 1, 2, \ldots$, such that $(m_2 - m_1)^{-1} < \varepsilon$, $2k_1(k_2 - k_1)^{-1} < \varepsilon$. Then for $N \geqslant \max\{m_i; k_i, i = 1, 2, m = 1, 2, \ldots, M, k = 1, 2, \ldots, K\}$ the equalities

$$z_\infty^\varepsilon(\psi^1, N) = \varphi_\varepsilon^1[y^N] = \frac{1}{K} \sum_{k=1}^{K} \frac{k_2 y_{k_1} - k_1 y_{k_2}}{k_2 - k_1}$$

$$z_\infty^\varepsilon(\psi^2, N) = \varphi_\varepsilon^2[y^N] = \frac{1}{M} \sum_{m=1}^{M} \frac{y_{m_2} - y_{m_1}}{m_2 - m_1}$$

yield strongly consistent, under the conditions being examined, estimates of the object's initial position and velocity, as

$$\rho_\infty^*[\varphi_\varepsilon^s] < \rho_\infty^*[\varphi^s] + \varepsilon, \quad s = 1, 2$$

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